General Certificate of Education
June 2009
Advanced Level Examination

MATHEMATICS **MFP2**
Unit Further Pure 2

Friday 5 June 2009 1.30 pm to 3.00 pm

For this paper you must have:
- an 8-page answer book
- the blue AQA booklet of formulae and statistical tables.

You may use a graphics calculator.

Time allowed: 1 hour 30 minutes

Instructions
- Use black ink or black ball-point pen. Pencil should only be used for drawing.
- Write the information required on the front of your answer book. The *Examining Body* for this paper is AQA. The *Paper Reference* is MFP2.
- Answer **all** questions.
- Show all necessary working; otherwise marks for method may be lost.

Information
- The maximum mark for this paper is 75.
- The marks for questions are shown in brackets.

Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.

Answer **all** questions.

1 Given that $z = 2e^{\frac{\pi i}{12}}$ satisfies the equation

$$z^4 = a(1 + \sqrt{3}\,i)$$

where a is real:

(a) find the value of a; *(3 marks)*

(b) find the other three roots of this equation, giving your answers in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$. *(5 marks)*

2 (a) Given that

$$\frac{1}{4r^2 - 1} = \frac{A}{2r - 1} + \frac{B}{2r + 1}$$

find the values of A and B. *(2 marks)*

(b) Use the method of differences to show that

$$\sum_{r=1}^{n} \frac{1}{4r^2 - 1} = \frac{n}{2n + 1}$$ *(3 marks)*

(c) Find the least value of n for which $\sum_{r=1}^{n} \frac{1}{4r^2 - 1}$ differs from 0.5 by less than 0.001. *(3 marks)*

3 The cubic equation

$$z^3 + pz^2 + 25z + q = 0$$

where p and q are real, has a root $\alpha = 2 - 3i$.

(a) Write down another non-real root, β, of this equation. *(1 mark)*

(b) Find:

(i) the value of $\alpha\beta$; *(1 mark)*

(ii) the third root, γ, of the equation; *(3 marks)*

(iii) the values of p and q. *(3 marks)*

4 (a) Sketch the graph of $y = \tanh x$. *(2 marks)*

(b) Given that $u = \tanh x$, use the definitions of $\sinh x$ and $\cosh x$ in terms of e^x and e^{-x} to show that

$$x = \frac{1}{2}\ln\left(\frac{1+u}{1-u}\right)$$ *(6 marks)*

(c) (i) Show that the equation

$$3\,\mathrm{sech}^2 x + 7\tanh x = 5$$

can be written as

$$3\tanh^2 x - 7\tanh x + 2 = 0$$ *(2 marks)*

(ii) Show that the equation

$$3\tanh^2 x - 7\tanh x + 2 = 0$$

has only one solution for x.

Find this solution in the form $\frac{1}{2}\ln a$, where a is an integer. *(5 marks)*

5 (a) Prove by induction that, if n is a positive integer,

$$(\cos\theta + \mathrm{i}\sin\theta)^n = \cos n\theta + \mathrm{i}\sin n\theta$$ *(5 marks)*

(b) Hence, given that

$$z = \cos\theta + \mathrm{i}\sin\theta$$

show that

$$z^n + \frac{1}{z^n} = 2\cos n\theta$$ *(3 marks)*

(c) Given further that $z + \frac{1}{z} = \sqrt{2}$, find the value of

$$z^{10} + \frac{1}{z^{10}}$$ *(4 marks)*

Turn over for the next question

Turn over ▶

6 (a) Two points, A and B, on an Argand diagram are represented by the complex numbers $2+3\mathrm{i}$ and $-4-5\mathrm{i}$ respectively. Given that the points A and B are at the ends of a diameter of a circle C_1, express the equation of C_1 in the form $|z-z_0|=k$. *(4 marks)*

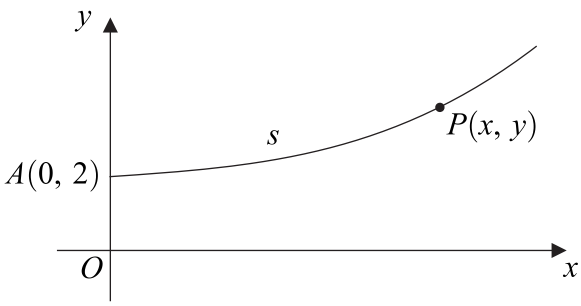
(b) A second circle, C_2, is represented on the Argand diagram by the equation $|z-5+4\mathrm{i}|=4$. Sketch on one Argand diagram both C_1 and C_2. *(3 marks)*

(c) The points representing the complex numbers z_1 and z_2 lie on C_1 and C_2 respectively and are such that $|z_1-z_2|$ has its maximum value. Find this maximum value, giving your answer in the form $a+b\sqrt{5}$. *(5 marks)*

7 The diagram shows a curve which starts from the point A with coordinates $(0, 2)$. The curve is such that, at every point P on the curve,

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\tfrac{1}{2}s$$

where s is the length of the arc AP.



(a) (i) Show that

$$\frac{\mathrm{d}s}{\mathrm{d}x}=\tfrac{1}{2}\sqrt{4+s^2}$$ *(3 marks)*

(ii) Hence show that

$$s=2\sinh\frac{x}{2}$$ *(4 marks)*

(iii) Hence find the cartesian equation of the curve. *(3 marks)*

(b) Show that

$$y^2=4+s^2$$ *(2 marks)*

END OF QUESTIONS